

EXTENDED SKEW-SYMMETRIC FORM FOR SUMMATION-BY-PARTS OPERATORS

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ABSTRACT. Summation-by-parts (SBP) operators originate in a finite difference framework for hyperbolic conservation laws. They can be used to prove conservation and discrete stability in a way similar to the derivation of continuous well-posedness. Recently, some extensions of these operators to other frameworks for high-order methods have been proposed. Here, the results of Ranocha, Öffner and Sonar (2015) on SBP operators for correction procedure via reconstruction are extended through a generalised analytical notion of SBP methods and a new form of correction terms, yielding conservation and stability in a discrete norm for a broad class of methods. Especially, SBP operators for nodal bases, possibly with dense norm and not including boundary points are considered.

1. INTRODUCTION

Many physical systems can be described by (systems of) hyperbolic conservation laws, e.g. fluid dynamics, electrodynamics or astrophysics. Computational methods are needed to obtain approximate solutions, but often low-order methods are used in computational fluid dynamics (CFD) and related areas, since they are simple, robust and reliable. On the other hand, high-order methods can be much more efficient, but are more complicated and less robust in general, the difficulty being the development of discontinuities in finite time even for smooth fluxes and initial data, which is characteristic for hyperbolic conservation laws.

Dealing with these discontinuities in the analytical approach relies on *entropy* conditions, inspired from physical principles. Thus, for numerical methods, entropy stability is crucial. Additionally, the conservation properties of the governing system must be retained.

Summation-by-parts (SBP) operators originate in the *finite difference* (FD) framework and yield an approach to prove stability in a way similar to the continuous investigations by mimicking *integration-by-parts* on a discrete level, see inter alia the review articles [SN14, FHZ14] and references cited therein. To get conservation and stability, (exterior and inter-element) boundary conditions are imposed weakly by *simultaneous approximation terms* (SATs) and skew-symmetric formulations for nonlinear conservation laws are used [FCN⁺13].

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Gassner [Gas13] applied the SBP framework to a *discontinuous Galerkin* (DG) spectral element method (DGSEM) using the nodes of Lobatto-Legendre quadrature. Additionally, Fernández et al. [FBZ14] proposed an extended definition of SBP methods in a numerical framework relying on nodal representations.

Ranocha et al. [RÖS15] investigated connections between SBP methods and the general framework of *correction procedure via reconstruction* (CPR), unifying the flux reconstruction Huynh [Huy07] and lifting collocation penalty [WG09] schemes. Several high-order methods such as DG, spectral volume and spectral difference methods can be formulated in this framework, see inter alia the review [HWV14] and references cited therein.

At first, we will give a brief review of SBP concepts and CPR methods in section 2, followed by an introduction to some of our results about diagonal-norm SBP CPR methods.

The main contribution of this article will be presented in section 3. There, a more general setting for SBP CPR methods in one dimension will be described. Based on this, an extended skew-symmetric form for nonlinear conservation laws is proposed using Burgers' equation as an example. Extending the correction terms in this form, conservation and nonlinear entropy stability are proved for general SBP CPR semidiscretisations, including both nodal bases without boundary nodes (e.g. Gauß-Legendre nodes) and modal bases (e.g. Legendre polynomials). Numerical examples are presented thereafter. Additionally, a brief comparison with the numerical setting of Fernández et al. [FBZ14] is given.

Finally, we summarise our results in section 4, followed by a discussion and further topics of research.

2. SKEW-SYMMETRIC FORM AND DIAGONAL-NORM SBP OPERATORS

In this section we repeat the basic concept and results about summation-by-parts operators for correction procedure via reconstruction of [RÖS15]. Both finite difference (FD) SBP methods and CPR schemes are designed as semidiscretisations of hyperbolic conservation laws

$$(1) \quad \partial_t u + \partial_x f(u) = 0,$$

equipped with appropriate initial and boundary conditions.

2.1. SBP CPR methods – basic concept. Traditionally, SBP operators are used in the FD framework. To compute the semidiscretisation of (1), $f(u)$ is evaluated at each node and a difference operator is applied. The notation using vectors \underline{u} for the solution values and the differentiation matrix \underline{D} is very common and results in a finite difference approximation $\underline{D}\underline{f}$ of $\partial_x f$. In order to be an SBP operator, the derivative matrix needs to be written as $\underline{D} = \underline{P}^{-1}\underline{Q}$, $\underline{Q} + \underline{Q}^T = \underline{B} = \text{diag}(-1, 0, \dots, 0, 1)$, where \underline{P} is a symmetric and positive definite matrix with associated norm $\|\underline{u}\|_P^2 = \underline{u}^T \underline{P} \underline{u}$, approximating the L^2 norm, see inter alia the review [SN14] and references cited therein.

Boundary (both of the computational domain and between blocks) conditions are imposed weakly, using a *simultaneous-approximation-term* (SAT) formulation (as described inter alia in [FCN⁺13]), involving differences of desired and given values at boundary points.

The FR approach in one space dimension described by Huynh [Huy07] uses a nodal polynomial basis of order p in the standard element $[-1, 1]$. All elements are mapped to this standard element and the computations are performed there. The semidiscretisation of (1) (i.e. the computation of $\partial_x f(u)$) consists of the following steps, see also the review [HWV14] and references cited therein:

- Interpolate the solution u to the cell boundaries at -1 and 1 (if these values are not already given as coefficients of the nodal basis).
- Compute common numerical fluxes f^{num} at each cell boundary.
- Compute the flux $f(u)$ pointwise in each node.
- Interpolate the flux $f(u)$ to the boundary and add polynomial correction functions g_L, g_R of degree $p + 1$, multiplied by the difference $f_{L/R} - f_{L/R}^{num}$ of the flux and the numerical flux at the corresponding boundary.
- Finally, compute the resulting derivative of $f + (f_L - f_L^{num})g_L + (f_R - f_R^{num})g_R$, using exact differentiation for the polynomial basis.

The lifting collocation penalty [WG09] was an extension of FR on triangles. Due to the tight connection between FR and LCP, both schemes are nowadays summarised as correction procedure via reconstruction (CPR).

[RÖS15] introduced a formulation of CPR methods with special attention paid to SBP operators.

After mapping each element to the standard element $[-1, 1]$, a CPR method can be formulated as

$$(2) \quad \partial_t \underline{u} + \underline{D} \underline{f} + \underline{C} (\underline{f}^{num} - \underline{R} \underline{f}) = 0.$$

Here, $\underline{u}, \underline{f}$ are the finite dimensional representation of $u, f(u)$ in the standard element and \underline{f}^{num} is the representation of the numerical flux on the boundary. The linear operators representing differentiation and restriction (interpolation) to the boundary of the standard element are represented via the matrices \underline{D} and \underline{R} , respectively. Other parameters of the correction operator are encoded in the correction matrix \underline{C} . Thus, for a given standard element, a CPR method is parametrised by

- A basis \mathcal{B} for the local expansion, determining the derivative and restriction (interpolation) matrices \underline{D} and \underline{R} .
- A correction matrix \underline{C} , adapted to the chosen basis.

For the representation of a diagonal-norm SBP operator, the basis \mathcal{B} has to be associated with a (volume) quadrature rule, given by nodes z_0, \dots, z_p and appropriate positive weights $\omega_0, \dots, \omega_p$. The values of u at the nodes are the coefficients of the local expansion, i.e. $\underline{u} = (u(z_0), \dots, u(z_p))^T$. The quadrature weights determine a positive definite Matrix $\underline{M} = \text{diag}(\omega_0, \dots, \omega_p)$ associated with a (discrete) norm $\|\underline{u}\|_M^2 = \underline{u}^T \underline{M} \underline{u}$. Besides the volume quadrature rule, there must be a quadrature rule for the boundary, approximating the outward flux through the boundary as in the divergence theorem. In the present one dimensional setting, this quadrature rule is simply given by exact evaluation at the endpoints $\cdot|_{-1}^1$. The basis and its associated quadrature rules must satisfy the SBP property

$$(3) \quad \underline{M} \underline{D} + \underline{D}^T \underline{M} = \underline{R}^T \underline{B} \underline{R},$$

in order to mimic integration by parts on a discrete level

$$\begin{aligned}
 & \underline{u}^T \underline{M} \underline{D} \underline{v} + (\underline{D} \underline{u})^T \underline{M} \underline{v} \\
 (4) \quad & \approx \int_{-1}^1 u \partial_x v \, dx + \int_{-1}^1 \partial_x u v \, dx = u v \Big|_{-1}^1 \\
 & \approx (\underline{R} \underline{u})^T \underline{B} (\underline{R} \underline{u}).
 \end{aligned}$$

As an example, consider Gauß-Lobatto-Legendre integration with its associated basis of point values at Lobatto nodes in $[-1, 1]$. Then, the restriction and boundary integral matrices reduce to

$$(5) \quad \underline{R} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the special choice $\underline{C} = \underline{M}^{-1} \underline{R}^T \underline{B}$ and defining $\tilde{\underline{B}} := \underline{R}^T \underline{B} \underline{R}$, i.e. $\tilde{\underline{B}} = \text{diag}(-1, 0, \dots, 0, 1)$, the CPR method of equation (2) reduces to

$$(6) \quad \partial_t \underline{u} + \underline{D} \underline{f} + \underline{M}^{-1} \tilde{\underline{B}} (\tilde{f}^{num} - \underline{f}) = 0,$$

where $\tilde{f}^{num} = (f_L^{num}, 0, \dots, 0, f_R^{num})$ contains the numerical flux at the left and right boundary and satisfies $\underline{f}^{num} = \underline{R} \tilde{f}^{num}$. Equation (6) is the strong form of the DGSEM formulation of Gassner [Gas13], which he proved to be a diagonal norm SBP operator.

2.2. Results about the skew-symmetric form and diagonal-norm SBP operators. Stability properties for linear and nonlinear problems can be very different. [RÖS15] considered Burgers' equation

$$(7) \quad \partial_t u + \partial_x \frac{u^2}{2} = 0$$

in one space dimension with periodic boundary conditions and appropriate initial condition. In this investigation, they employed the same ansatz as Gassner [Gas13]. First, they analysed for Lobatto-Legendre nodes including the boundaries the skew-symmetric form

$$(8) \quad \partial_t \underline{u} = -\frac{1}{2} \underline{D} \underline{u}^2 - (1 - \alpha) (\underline{u} \underline{D} \underline{u} - \frac{1}{2} \underline{D} \underline{u}^2) - \underline{C} (\underline{f}^{num} - \frac{1}{2} \underline{R} \underline{u}^2),$$

where $\underline{u} = \text{diag}(u)$ denotes the matrix representing multiplication with u . With $\alpha = \frac{2}{3}$ and the condition

$$(9) \quad \frac{1}{6} (u_-^3 - u_+^3) - (u_- - u_+) f^{num}(u_-, u_+) \leq 0,$$

on the numerical flux, they proved stability for the CPR method (8) [RÖS15, Lemma 6].

For a general SBP basis the stability investigation is more complicated. [RÖS15] introduced a further correction term. They considered and analysed an SBP CPR method with corrected divergence (skew-symmetric form) and corrected boundary terms

$$(10) \quad \partial_t \underline{u} + \frac{\alpha}{2} \underline{D} \underline{u}^2 + (1 - \alpha) \underline{u} \underline{D} \underline{u} + \underline{C} \left(\underline{f}^{num} - \frac{\beta}{2} \underline{R} \underline{u}^2 - \frac{1 - \beta}{2} (\underline{R} \underline{u})^2 \right) = 0,$$

$0 \leq \alpha, \beta \leq 1$. Finally, they proved the following result:

Theorem 1. *If the numerical flux f^{num} satisfies*

$$(11) \quad \frac{1}{6}(u_-^3 - u_+^3) - (u_- - u_+)f^{num}(u_-, u_+) \leq 0,$$

then a (diagonal-norm) SBP CPR method

$$(12) \quad \partial_t \underline{u} + \underline{D} \frac{1}{2} \underline{u}^2 + \underline{c}_{div} + \underline{C} \left(f^{num} - \underline{R} \frac{1}{2} \underline{u}^2 - \underline{c}_{res} \right) = 0,$$

with $\underline{C} = \underline{M}^{-1} \underline{R}^T \underline{B}$ and correction terms for both divergence and restriction to the boundary

$$(13) \quad \underline{c}_{div} = \frac{1}{3} \left(\underline{u} \underline{D} \underline{u} - \frac{1}{2} \underline{D} \underline{u} \underline{u} \right), \quad \underline{c}_{res} = \frac{1}{6} \left((\underline{R} \underline{u})^2 - \underline{R} \underline{u} \underline{u} \right),$$

for the inviscid Burgers' equation (7) is both conservative and stable in the discrete norm $\|\cdot\|_M$ induced by \underline{M} . Numerical fluxes fulfilling this condition are inter alia

- *the energy conservative (ECON) flux*

$$(14) \quad f^{num}(u_-, u_+) = \frac{1}{4}(u_+^2 + u_-^2) - \frac{(u_+ - u_-)^2}{12}$$

- *the local Lax-Friedrichs (LLF) flux*

$$(15) \quad f^{num}(u_-, u_+) = \frac{1}{4}(u_+^2 + u_-^2) - \frac{\max(|u_+|, |u_-|)}{2}(u_+ - u_-)$$

- *and Osher's flux*

$$(16) \quad f^{num}(u_-, u_+) = \begin{cases} \frac{u_-^2}{2}, & u_+, u_- > 0, \\ \frac{u_+^2}{2}, & u_+, u_- < 0, \\ \frac{u_+^2}{2} + \frac{u_-^2}{2}, & u_- \geq 0 \geq u_+, \\ 0, & u_- \leq 0 \leq u_+. \end{cases}$$

3. ABSTRACT VIEW AND GENERALISATION

The basic setting described in [RÖS15] uses diagonal norm SBP operators and nodal bases, associated with quadrature rules with positive weights. These operators have been used in the context of CPR methods to obtain conservative and stable semidiscretisations for linear advection and Burgers' equation. This chapter provides a more abstract view on the results and generalised schemes with a new form of the correction terms, allowing both modal and nodal basis with arbitrary (dense) norm.

3.1. Analytical setting in one dimension. Continuing the investigations of [RÖS15], an analytical setting in the one-dimensional standard element $\Omega = [-1, 1]$ is presented at first. The semidiscretisation in space consists of the representation of a numerical solution in a (real) finite dimensional Hilbert space X_V . Hitherto, X_V has been the space of polynomials of degree $\leq p$, i.e. $\dim X_V = p + 1$. X_V is equipped with a suitable basis \mathcal{B}_V , e.g. a Lagrange (interpolation) basis for Gauß-Legendre or Lobatto-Legendre quadrature nodes. With regard to \mathcal{B}_V , the *scalar product and associated norm* on X_V are given by a symmetric and positive-definite matrix \underline{M} , approximating the L^2 norm on X_V , i.e.

$$(17) \quad \underline{u}^T \underline{M} \underline{v} = \langle \underline{u}, \underline{v} \rangle_M \approx \int_{\Omega} uv = \langle u, v \rangle_{L^2}.$$

In one dimension, a *divergence* (derivative) operator mapping X_V to X_V is represented by a matrix $\underline{\underline{D}}$.

Besides X_V , the vector space of functions on the (one-dimensional) *volume* Ω , a vector space X_B of functions on the (0-dimensional) *boundary* $\partial\Omega$ of the standard element Ω with its associated basis \mathcal{B}_B has to be considered. In the simple one-dimensional case, X_B is a two-dimensional vector space and \mathcal{B}_B is chosen to represent point values at -1 and 1 . On the boundary, a bilinear form is represented by a matrix $\underline{\underline{B}}$, approximating the *boundary (surface) integral* in the outward normal direction, i.e. evaluation at the boundary. More precisely, $\underline{\underline{B}}$ maps $X_B \times X_B$ to \mathbb{R} and

$$(18) \quad \underline{u}_B^T \underline{\underline{B}} \underline{f}_B = B(u_B, f_B) \approx u_B f_B \Big|_{-1}^1.$$

In the simple one-dimensional setting, u_B and f_B are both scalar functions and $\int_{\partial\Omega} u_B f_B \cdot n = u(1)f(1) - u(-1)f(-1)$, i.e. $\underline{\underline{B}} = \text{diag}(-1, 1)$ if \mathcal{B}_B is ordered such that the value at -1 is the first coefficient. With regard to the chosen bases \mathcal{B}_V and \mathcal{B}_B , a *restriction* operator is represented by a matrix $\underline{\underline{R}}$, mapping a function u on the volume to its values at the boundary. The SBP property mimics integration by parts and requires

$$(19) \quad \underline{\underline{M}} \underline{\underline{D}} + \underline{\underline{D}}^T \underline{\underline{M}} = \underline{\underline{R}}^T \underline{\underline{B}} \underline{\underline{R}}.$$

A CPR method is further parametrised by a *correction* or *penalty* operator, represented by a matrix $\underline{\underline{C}}$ adapted to the chosen bases. The canonical choice is $\underline{\underline{C}} = \underline{\underline{M}}^{-1} \underline{\underline{R}}^T \underline{\underline{B}}$ as described in [RÖS15], especially for nonlinear equations. For linear advection, other choices of $\underline{\underline{C}}$ are possible, recovering the full range of linearly stable schemes presented in [VFWJ15], see [RÖS15, section 3].

Since nonlinear fluxes $f(u)$ appear and are of interest, nonlinear operations on X_V have to be described. In general, if X_V is a finite-dimensional vector space of polynomials containing polynomials of degree $\leq p$ ($p \geq 1$ and p is minimal), then the product of $u, v \in X_V$ is a polynomial of degree $\leq 2p$, i.e. not in X_V in general. Therefore, discrete multiplication is not exact. Thus, multiplying $v \in X_V$ with $u \in X_V$ yields $\underline{u}^+ \underline{v} \in X_V^+$, where $X_V^+ \supset X_V$ is a vector space of higher dimension. After this exact multiplication, a *projection* on X_V is performed, resulting in $\underline{u} \underline{v} \in X_V$. For a nodal basis \mathcal{B}_V , the natural projection is given by pointwise evaluation at the nodes. However, for a modal basis of Legendre polynomials, the natural projection is an L^2 orthogonal projection on X_V . Disappointingly, this concept does not easily extend to division, since L^2 projection of rational functions is not a simple task.

We have already extended this analytical setting to multiple dimensions with some minor modifications. Currently, we are working on the implementation and will publish our results in a forthcoming paper.

3.2. Revisiting Burgers' equation. Investigating again a skew-symmetric SBP CPR method without the assumption of a nodal and/or orthogonal basis, some further complications arise. In contrast to the manipulations used to prove Theorem 9 of [RÖS15] (see also [Gas13]), \underline{u} and \underline{M} might not commute, either because the nodal basis is not orthogonal or because a modal basis is chosen. Therefore, the correction terms for the divergence and restriction

$$((13)) \quad \underline{c}_{div} = \frac{1}{3} \left(\underline{u} \underline{\underline{D}} \underline{u} - \frac{1}{2} \underline{\underline{D}} \underline{u} \underline{u} \right), \quad \underline{c}_{res} = \frac{1}{6} \left((\underline{\underline{R}} \underline{u})^2 - \underline{\underline{R}} \underline{u} \underline{u} \right),$$

do not suffice to prove conservation and stability. The reason is again inexactness of discrete multiplication. A multiplication operator \underline{u} should be self-adjoint, at least in a finite-dimensional space (and in general, if a correct domain is chosen). Thus, instead of \underline{u} in the first term of \underline{c}_{div} , the adjoint \underline{u}^* of \underline{u} with respect to the scalar product induced by \underline{M} is proposed. The symmetry condition

$$(20) \quad \langle \underline{v}, \underline{u} \underline{w} \rangle_M = \langle \underline{u}^* \underline{v}, \underline{w} \rangle_M$$

can be written as

$$(21) \quad \underline{v}^T \underline{M} \underline{u} \underline{w} = \underline{v}^T (\underline{u}^*)^T \underline{M} \underline{w}.$$

Thus, since \underline{v} and \underline{w} are arbitrary, $\underline{M} \underline{u} = (\underline{u}^*)^T \underline{M}$, i.e. $\underline{u}^* = \underline{M}^{-1} \underline{u}^T \underline{M}$, and the generalised correction terms are

$$(22) \quad \underline{c}_{div} = \frac{1}{3} \left(\underline{M}^{-1} \underline{u}^T \underline{M} \underline{D} \underline{u} - \frac{1}{2} \underline{D} \underline{u} \underline{u} \right), \quad \underline{c}_{res} = \frac{1}{6} \left((\underline{R} \underline{u})^2 - \underline{R} \underline{u} \underline{u} \right).$$

Using these correction terms, Theorem 9 of [RÖS15] is generalised by

Theorem 2. *If the numerical flux f^{num} satisfies*

$$(23) \quad \frac{1}{6} (u_-^3 - u_+^3) - (u_- - u_+) f^{num}(u_-, u_+) \leq 0,$$

then a general SBP CPR method with $\underline{C} = \underline{M}^{-1} \underline{R}^T \underline{B}$ and correction terms (22) for both divergence and restriction to the boundary

$$(24) \quad \partial_t \underline{u} + \underline{D} \frac{1}{2} \underline{u}^2 + \underline{c}_{div} + \underline{C} \left(f^{num} - \underline{R} \frac{1}{2} \underline{u}^2 - \underline{c}_{res} \right) = 0,$$

for the inviscid Burgers' equation (7) is both conservative and stable in the discrete norm $\|\cdot\|_M$ induced by \underline{M} . Numerical fluxes fulfilling this condition are inter alia

- *the energy conservative (ECON) flux (14),*
- *the local Lax-Friedrichs (LLF) flux (15),*
- *and Osher's flux (16).*

Proof. Multiplying $\partial_t \underline{u}$ with $\underline{v}^T \underline{M}$, inserting $\underline{C} = \underline{M}^{-1} \underline{R}^T \underline{B}$ and applying the SBP property (19) yields

$$(25) \quad \begin{aligned} \underline{v}^T \underline{M} \partial_t \underline{u} &= -\frac{1}{2} \underline{v}^T \underline{M} \underline{D} \underline{u} \underline{u} - \underline{v}^T \underline{M} \underline{c}_{div} - \underline{v}^T \underline{R}^T \underline{B} \left(f^{num} - \frac{1}{2} \underline{R} \underline{u} \underline{u} - \underline{c}_{res} \right) \\ &= +\frac{1}{2} \underline{v}^T \underline{D}^T \underline{M} \underline{u} \underline{u} - \frac{1}{2} \underline{v}^T \underline{R}^T \underline{B} \underline{R} \underline{u} \underline{u} \\ &\quad - \underline{v}^T \underline{M} \underline{c}_{div} - \underline{v}^T \underline{R}^T \underline{B} \left(f^{num} - \frac{1}{2} \underline{R} \underline{u} \underline{u} - \underline{c}_{res} \right). \end{aligned}$$

Gathering terms and inserting \underline{c}_{div} , \underline{c}_{res} from equation (22) results in

$$(26) \quad \begin{aligned} \underline{v}^T \underline{M} \partial_t \underline{u} &= \frac{1}{2} \underline{v}^T \underline{D}^T \underline{M} \underline{u} \underline{u} - \underline{v}^T \underline{M} \underline{c}_{div} - \underline{v}^T \underline{R}^T \underline{B} f^{num} + \underline{v}^T \underline{R}^T \underline{B} \underline{c}_{res} \\ &= \frac{1}{2} \underline{v}^T \underline{D}^T \underline{M} \underline{u} \underline{u} - \frac{1}{3} \underline{v}^T \underline{u}^T \underline{M} \underline{D} \underline{u} + \frac{1}{6} \underline{v}^T \underline{M} \underline{D} \underline{u} \underline{u} \\ &\quad - \underline{v}^T \underline{R}^T \underline{B} f^{num} + \frac{1}{6} \underline{v}^T \underline{R}^T \underline{B} (\underline{R} \underline{u})^2 - \frac{1}{6} \underline{v}^T \underline{R}^T \underline{B} \underline{R} \underline{u} \underline{u}. \end{aligned}$$

Applying the SBP property (19) for the third term yields

$$\begin{aligned}
 \underline{v}^T \underline{M} \partial_t \underline{u} &= \frac{1}{2} \underline{v}^T \underline{D}^T \underline{M} \underline{u} \underline{u} - \frac{1}{3} \underline{v}^T \underline{u}^T \underline{M} \underline{D} \underline{u} + \frac{1}{6} \underline{v}^T \underline{R}^T \underline{B} \underline{R} \underline{u} \underline{u} - \frac{1}{6} \underline{v}^T \underline{D}^T \underline{M} \underline{u} \underline{u} \\
 &\quad - \underline{v}^T \underline{R}^T \underline{B} \underline{f}^{num} + \frac{1}{6} \underline{v}^T \underline{R}^T \underline{B} (\underline{R} \underline{u})^2 - \frac{1}{6} \underline{v}^T \underline{R}^T \underline{B} \underline{R} \underline{u} \underline{u} \\
 &= \frac{1}{3} \underline{v}^T \underline{D}^T \underline{M} \underline{u} \underline{u} - \frac{1}{3} \underline{v}^T \underline{u}^T \underline{M} \underline{D} \underline{u} - \underline{v}^T \underline{R}^T \underline{B} \underline{f}^{num} + \frac{1}{6} \underline{v}^T \underline{R}^T \underline{B} (\underline{R} \underline{u})^2.
 \end{aligned}
 \tag{27}$$

In order to obtain *stability*, $\frac{1}{2} \frac{d}{dt} \|\underline{u}\|_M^2 = \underline{u}^T \underline{M} \partial_t \underline{u}$ has to be considered. Thus, setting $\underline{v} = \underline{u}$ in (27) and using the symmetry of the \underline{M} results in

$$\frac{1}{2} \frac{d}{dt} \|\underline{u}\|_M^2 = -\underline{u}^T \underline{R}^T \underline{B} \underline{f}^{num} + \frac{1}{6} \underline{u}^T \underline{R}^T \underline{B} (\underline{R} \underline{u})^2,
 \tag{28}$$

Denoting the values from the cells left and right to a given boundary node with indices $-$ and $+$, respectively, and summing over all elements, the contribution of one boundary node to $\frac{1}{2} \frac{d}{dt} \|\underline{u}\|_M^2$ is

$$\frac{1}{6} (\underline{u}_-^3 - \underline{u}_+^3) - (\underline{u}_- - \underline{u}_+) \underline{f}^{num} (\underline{u}_-, \underline{u}_+).
 \tag{29}$$

If this is non-positive (as assumed), stability in the discrete norm described by \underline{M} is guaranteed.

Investigation *conservation* by setting $\underline{v} = \underline{1}$ in (27), using $\underline{D} \underline{1} = 0$ (i.e. exact differentiation for constant functions) and $\underline{u} \underline{1} = \underline{u}$ (i.e. exact multiplication with constant functions) yields

$$\frac{d}{dt} \underline{1}^T \underline{M} \underline{u} = -\frac{1}{3} \underline{u}^T \underline{M} \underline{D} \underline{u} - \underline{1}^T \underline{R}^T \underline{B} \underline{f}^{num} + \frac{1}{6} \underline{1}^T \underline{R}^T \underline{B} (\underline{R} \underline{u})^2,
 \tag{30}$$

Rewriting the first term (by the SBP property (19)) as

$$-\frac{1}{3} \underline{u}^T \underline{M} \underline{D} \underline{u} = -\frac{1}{6} \underline{u}^T \underline{M} \underline{D} \underline{u} + \frac{1}{6} \underline{u}^T \underline{D}^T \underline{M} \underline{u} - \frac{1}{6} \underline{u}^T \underline{R}^T \underline{B} \underline{R} \underline{u} = -\frac{1}{6} \underline{u}^T \underline{R}^T \underline{B} \underline{R} \underline{u}
 \tag{31}$$

results in

$$\frac{d}{dt} \underline{1}^T \underline{M} \underline{u} = -\frac{1}{6} \underline{u}^T \underline{R}^T \underline{B} \underline{R} \underline{u} - \underline{1}^T \underline{R}^T \underline{B} \underline{f}^{num} + \frac{1}{6} \underline{1}^T \underline{R}^T \underline{B} (\underline{R} \underline{u})^2.
 \tag{32}$$

Denoting the values of u at the left and right boundary as u_L and u_R , respectively,

$$\underline{u}^T \underline{R}^T \underline{B} \underline{R} \underline{u} = u_R \cdot u_R - u_L \cdot u_L = 1 \cdot u_R^2 - 1 \cdot u_L^2 = \underline{1}^T \underline{R}^T \underline{B} (\underline{R} \underline{u})^2.
 \tag{33}$$

and therefore

$$\frac{d}{dt} \underline{1}^T \underline{M} \underline{u} = -\underline{1}^T \underline{R}^T \underline{B} \underline{f}^{num}.
 \tag{34}$$

Thus, summing over all elements, the contribution of both cells sharing a common boundary node cancel each other, since the numerical flux \underline{f}^{num} is the same for both elements. \square

3.3. Numerical results for dense norm and modal bases. Aside from the basis, the parameters of [RÖS15, section 4.4] are used to obtain numerical solutions of Burgers' equation (7) in the domain $[0, 2]$ with periodic boundary conditions. The initial condition

$$u(0, x) = u_0(x) = \sin(\pi x) + 0.01
 \tag{35}$$

is evolved in time using the classical fourth order Runge-Kutta method with 10,000 time steps in the time interval $[0, 3]$. As semidiscretisation in space, several SBP CPR methods with $N = 20$ equally spaced elements describing polynomials of degree $\leq p = 7$ and correction terms (22) are used.

As nodal bases with diagonal norm matrix $\underline{\underline{M}}$, the nodes of Gauß-Legendre and Lobatto-Legendre quadrature rules are used. The new nodal bases represent polynomials of degree $\leq p = 7$ using their values at the

- roots $\xi_i = \cos \frac{(2i+1)\pi}{2p+2}$, $i = 0, \dots, p$,
- extrema $\xi_i = \cos \frac{i\pi}{p}$, $i = 0, \dots, p$

of Chebyshev polynomial T_{p+1} of first kind or the

- roots $\xi_i = \cos \frac{(i+1)\pi}{p+2}$, $i = 0, \dots, p$

of the Chebyshev polynomial U_{p+1} of second kind. The differentiation and norm matrices $\underline{\underline{D}}$, $\underline{\underline{M}}$ are computed via their representation for Legendre polynomials and a basis transformation using the associated Vandermonde matrix, see A. Multiplication is conducted pointwise at the corresponding Chebyshev nodes. For these bases, $\underline{\underline{M}}$ is not diagonal and multiplication operators \underline{u} are not $\underline{\underline{M}}$ -self-adjoint in general.

Additionally, a modal basis of Legendre polynomials is used, performing exact multiplication followed by an orthogonal projection. For this orthogonal basis, a multiplication operator \underline{u} is in general not diagonal, but $\underline{\underline{M}}$ -self-adjoint, as the following calculation for arbitrary polynomials u, v, w of degree $\leq p$ shows:

$$\begin{aligned}
 \langle \underline{v}, \underline{u} \underline{w} \rangle_M &= \underline{v}^T \underline{\underline{M}} \underline{u} \underline{w} = \int v \operatorname{proj}(u w) = \int v u w \\
 &= \int \operatorname{proj}(u v) w = \underline{v}^T \underline{u}^T \underline{\underline{M}} \underline{w} = \langle \underline{u} \underline{v}, \underline{w} \rangle_M.
 \end{aligned}
 \tag{36}$$

The third and fourth equality follow from the orthogonality of Legendre polynomials. Thus, multiplication operators \underline{u} are $\underline{\underline{M}}$ -self-adjoint.

An interpolation approach to compute the initial values for a Legendre basis using the nodes of all nodal bases presented in Figure 1 has been used. There is no visual difference between results for these different sets of nodes. In the following, interpolation via Gauß-Legendre nodes has been used.

The results of the computations using the local Lax-Friedrichs flux are shown in Figures 1 and 2. For comparison, the results of [RÖS15] using Gauß-Legendre and Lobatto-Legendre bases are included in the first rows. The values of $u(3)$ are in general similar – two approximately affine-linear parts and a discontinuous part with oscillations around $x = 1$. Despite of this, the intensity of oscillations depends on the bases and associated projection used for multiplication.

In this case, the roots of Chebyshev polynomials of second kind seem to perform worst, whereas Gauß-Legendre nodes and modal Legendre polynomials seem to be least oscillatory and visually indistinguishable. Contrary, the computations using a nodal basis are much more efficient, since only simple multiplication of nodal values has to be performed.

As expected, momentum is conserved for all bases and the discrete energy (entropy) is constant until $t \approx 0.5$ and decays afterwards, as can be seen in Figure 2.

These results are obtained using general SBP CPR methods (24) with both correction terms for divergence and restriction (22). Ignoring a non-trivial correction

term for a nodal basis leads to physically useless results, as shown for example by [RÖS15, Figure 11]. Results without the skew-symmetric correction \underline{c}_{div} are not plotted here. Additionally, the correction term \underline{c}_{div} using the \underline{M} -adjoint multiplication operator is verified numerically, since using the simple multiplication as in the previous chapter gives erroneous results, again not shown here.

Remarkably, the results (not plotted here) using a modal Legendre basis and either both or no correction term (\underline{c}_{div} , \underline{c}_{res}) are visually indistinguishable. Additionally, using only \underline{c}_{res} yields the same results. Contrary, using only a correction for the divergence results in varying momentum and physically useless results. Using an exact orthogonal projection during multiplication seems to be a good idea, but an analytical investigation of this phenomenon remains an open problem.

3.4. A brief view on a numerical setting. The analytical setting of section 3.1 is based on a given solution space X_V for the one-dimensional standard element, since the investigations in this work started from CPR methods, extending DG methods, which are also described by a fundamental basis. Contrary, the theory of SBP operators originates in FD methods, classically not equipped with a solution basis other than the nodal values. Nevertheless, Gassner [Gas13] adapted the SBP framework to a DGSEM with nodal Lobatto-Legendre basis and lumped mass matrix. Additionally, Fernández et al. [FBZ14] proposed a generalised SBP framework in one dimension based on nodal values without an analytical basis. Instead, the operators are required to fulfil the SBP property and some accuracy conditions, i.e. they should be exact for polynomials up to some degree $p \geq 1$. These ideas were extended by Hicken et al. [HFZ15] to multi-dimensional operators, focussing on diagonal-norm SBP operators on simplex elements in two and three dimensions, i.e. triangles and tetrahedra.

These extensions were applied to linear advection with constant velocity and proved to be conservative and stable in the norm associated with the SBP operator. Relaxing accuracy conditions potentially results in additional free parameters, allowing the construction of specialised schemes for different purposes. As already proved in [HZ13], SBP operators are tightly coupled to quadrature rules. Thus, different quadrature rules can be used to obtain SBP operators and vice versa.

All investigations conducted in the previous chapters and sections directly extend to these generalised FD SBP operators with diagonal or dense norm, respectively. Additionally, since these operators are described by the same matrices used hitherto in the investigations, they can be simply plugged in the numerical method for the calculations – up to the last step. In the analytical setting, the solution is completely determined by the given coefficients with regard to the chosen basis, i.e. sub-cell resolution of arbitrary accuracy is given. Especially, the solution can be plotted exactly as it is used in the computations. Contrary, using only nodal values at a given set of points without an interpretation as coefficients of a known basis, only these point values can be plotted as output seriously. Performing any interpolation would be a guess, but can in general not describe the solution accurately. From the authors' point of view, this is a drawback of the numerical setting without a basis as foundation. The inability to describe a modal basis does not seem to be equally unfavourable, since computing a correct orthogonal projection for division is not a straightforward task and nodal methods are much more efficient regarding evaluation times for nonlinear operations.

A solution of the interpolation problem would be to construct a basis describing a given SBP operator. For example, Gassner [Gas13] constructed a basis for a specially chosen FD SBP operator. However, there does not seem to be a straightforward way to construct such a basis in general.

4. SUMMARY AND DISCUSSION

In this work, an extended analytical framework for SBP methods has been proposed, extending the results of [RÖS15]. Thus, the linearly stable CPR methods of [VCJ11, VFWJ15] and the DGSEM of [Gas13] are embedded in this framework. Additionally, extended correction terms for nonlinear conservation laws are developed, using the inviscid Burgers' equation as an example. These correction terms for both divergence and restriction to the boundary extend the skew-symmetric form of conservation laws used in traditional FD SBP methods and the DGSEM based on Lobatto nodes [FCN⁺13, Gas13]. These new corrections allow for both modal and nodal SBP bases without any further conditions on the norm (e.g. diagonal) or the presence of nodes at the boundary. Using the SBP property, both conservation and stability in a discrete norm adapted to the chosen bases are proved. These results extend directly to traditional SBP methods lacking the foundation of an analytical basis, since only structural properties of the representations in a given basis are used.

We have already extended the analytical setting to multiple space dimensions, showing similarities with the numerical setting of [FBZ14, HFZ15], and are currently working on the implementation. This genuinely multi-dimensional formulation allows inter alia simplex elements and does not rely on a tensor product extension. Of course, the standard multi-dimensional setting using tensor products is embedded therein.

Other directions of research include fully discrete schemes, since a straightforward extension using an explicit time integrator does not seem to be provably stable. Additionally, the idea of the CPR framework, i.e. using another norm not associated directly with the given basis, will be further considered. Of course, other examples for nonlinear (systems of) conservation laws will also be investigated using the new ideas.

APPENDIX A. SOME BASES

To compute the matrices $\underline{\underline{M}}, \underline{\underline{D}}$ for the nodal bases using Chebyshev points, the associated matrices in a modal Legendre basis are used. The coordinate transformation from a nodal basis with nodes ξ_0, \dots, ξ_p to a modal basis of Legendre polynomials ϕ_0, \dots, ϕ_p of degree $\leq p$ is given by the Vandermonde matrix $\underline{\underline{V}}$ with $V_{i,j} = \phi_j(\xi_i)$. Writing vectors and matrices with regard to the modal basis with $\hat{\cdot}$, the transformation is $\underline{\underline{V}} \hat{\underline{\underline{u}}} = \underline{\underline{u}}$. Thus, operators like the derivative are transformed as $\hat{\underline{\underline{D}}} = \underline{\underline{V}}^{-1} \underline{\underline{D}} \underline{\underline{V}}$ and matrices associated with a scalar product like $\underline{\underline{M}}$ as $\hat{\underline{\underline{M}}} = \underline{\underline{V}}^T \underline{\underline{M}} \underline{\underline{V}}$.

The modal matrices are

$$(37) \quad \underline{\underline{\hat{M}}} = \begin{pmatrix} 2 & & & & \\ & \frac{2}{3} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{2}{2p+1} \end{pmatrix}, \quad \underline{\underline{\hat{D}}} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 3 & 0 & 3 & \dots \\ 0 & 0 & 0 & 5 & 0 & \dots \\ 0 & 0 & 0 & 0 & 7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Using $p = 2$ as an example, the nodal bases with dense norm are given by the following matrices.

- The roots of the Chebyshev polynomials of first kind are $\xi_i = \cos\left(\frac{2i+1}{2p+2}\pi\right)$, for $i = 0, \dots, p$. The Vandermonde matrix using 64 bit floating point numbers is approximately

$$(38) \quad \underline{\underline{V}} = \begin{pmatrix} 1.0 & 0.866\,025\,403\,784\,438\,7 & 0.625 \\ 1.0 & 6.123\,233\,995\,736\,766 \times 10^{-17} & -0.5 \\ 1.0 & -0.866\,025\,403\,784\,438\,7 & 0.625 \end{pmatrix}.$$

Calculating the mass matrix as $\underline{\underline{M}} = \underline{\underline{V}}^{-T} \underline{\underline{\hat{M}}} \underline{\underline{V}}$ results in

$$(39) \quad \underline{\underline{M}} = \begin{pmatrix} 0.399\,999\,999\,999\,999\,9 & 0.088\,888\,888\,888\,888\,71 & -0.044\,444\,444\,444\,444\,45 \\ 0.088\,888\,888\,888\,888\,8 & 0.933\,333\,333\,333\,333\,3 & 0.088\,888\,888\,888\,888\,96 \\ -0.044\,444\,444\,444\,444\,37 & 0.088\,888\,888\,888\,888\,99 & 0.399\,999\,999\,999\,999\,97 \end{pmatrix}.$$

The restriction (interpolation to the boundary) and boundary matrices used are

$$(40) \quad \underline{\underline{R}} = \begin{pmatrix} 0.089\,316\,397\,477\,040\,87 & -0.333\,333\,333\,333\,333\,2 & 1.244\,016\,935\,856\,292\,2 \\ 1.244\,016\,935\,856\,292\,2 & -0.333\,333\,333\,333\,333\,2 & 0.089\,316\,397\,477\,040\,82 \end{pmatrix},$$

$$(41) \quad \underline{\underline{B}} = \begin{pmatrix} -1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}.$$

Computing the derivative matrix via $\underline{\underline{D}} = \underline{\underline{V}} \underline{\underline{\hat{D}}} \underline{\underline{V}}^{-1}$ yields

$$(42) \quad \underline{\underline{D}} = \begin{pmatrix} 1.732\,050\,807\,568\,877\,2 & -2.309\,401\,076\,758\,503 & 0.577\,350\,269\,189\,625\,6 \\ 0.577\,350\,269\,189\,625\,7 & -1.632\,862\,398\,863\,137\,5 \times 10^{-16} & -0.577\,350\,269\,189\,625\,6 \\ -0.577\,350\,269\,189\,625\,7 & 2.309\,401\,076\,758\,503 & -1.732\,050\,807\,568\,877\,2 \end{pmatrix}.$$

- The extrema of the Chebyshev polynomials of first kind are $\xi_i = \cos\left(\frac{i}{p}\pi\right)$, for $i = 0, \dots, p$. Thus, the matrices are

$$(43) \quad \underline{\underline{V}} = \begin{pmatrix} 1.0 & 1.0 & 1.0 \\ 1.0 & 6.123\,233\,995\,736\,766 \times 10^{-17} & -0.5 \\ 1.0 & -1.0 & 1.0 \end{pmatrix},$$

$$(44) \quad \underline{\underline{M}} = \begin{pmatrix} 0.266\,666\,666\,666\,666\,66 & 0.133\,333\,333\,333\,333\,25 & -0.066\,666\,666\,666\,666\,65 \\ 0.133\,333\,333\,333\,333\,3 & 1.066\,666\,666\,666\,666\,4 & 0.133\,333\,333\,333\,333\,47 \\ -0.066\,666\,666\,666\,666\,68 & 0.133\,333\,333\,333\,333\,44 & 0.266\,666\,666\,666\,666\,7 \end{pmatrix},$$

$$(45) \quad \underline{\underline{R}} = \begin{pmatrix} 0.0 & 0.0 & 1.0 \\ 1.0 & 0.0 & 0.0 \end{pmatrix},$$

$$(46) \quad \underline{\underline{D}} = \begin{pmatrix} 1.500\,000\,000\,000\,000\,2 & -2.0 & 0.499\,999\,999\,999\,999\,8 \\ 0.500\,000\,000\,000\,000\,1 & -1.224\,646\,799\,147\,353 \times 10^{-16} & -0.499\,999\,999\,999\,999\,94 \\ -0.500\,000\,000\,000\,000\,2 & 2.0 & -1.499\,999\,999\,999\,999\,8 \end{pmatrix}.$$

- Finally, the roots of the Chebyshev polynomials of second kind are $\xi_i = \cos\left(\frac{i+1}{p+2}\pi\right)$, for $i = 0, \dots, p$. Therefore, the matrices are

$$(47) \quad \underline{\underline{V}} = \begin{pmatrix} 1.0 & 0.707\,106\,781\,186\,547\,6 & 0.250\,000\,000\,000\,000\,1 \\ 1.0 & 6.123\,233\,995\,736\,766 \times 10^{-17} & -0.5 \\ 1.0 & -0.707\,106\,781\,186\,547\,5 & 0.249\,999\,999\,999\,999\,9 \end{pmatrix},$$

$$(48) \quad \underline{\underline{M}} = \begin{pmatrix} 0.733\,333\,333\,333\,333\,2 & -0.133\,333\,333\,333\,333\,3 & 0.066\,666\,666\,666\,666\,79 \\ -0.133\,333\,333\,333\,333\,3 & 0.933\,333\,333\,333\,333\,2 & -0.133\,333\,333\,333\,333\,3 \\ 0.066\,666\,666\,666\,666\,79 & -0.133\,333\,333\,333\,333\,3 & 0.733\,333\,333\,333\,333\,3 \end{pmatrix},$$

$$(49) \quad \underline{\underline{R}} = \begin{pmatrix} 0.292\,893\,218\,813\,452\,6 & -1.000\,000\,000\,000\,000\,2 & 1.707\,106\,781\,186\,547\,7 \\ 1.707\,106\,781\,186\,547\,7 & -0.999\,999\,999\,999\,999\,9 & 0.292\,893\,218\,813\,452\,4 \end{pmatrix},$$

$$(50) \quad \underline{\underline{D}} = \begin{pmatrix} 2.121\,320\,343\,559\,643 & -2.828\,427\,124\,746\,19 & 0.707\,106\,781\,186\,547\,2 \\ 0.707\,106\,781\,186\,547\,5 & -3.558\,369\,867\,163\,396 \times 10^{-17} & -0.707\,106\,781\,186\,547\,5 \\ -0.707\,106\,781\,186\,547\,9 & 2.828\,427\,124\,746\,19 & -2.121\,320\,343\,559\,642 \end{pmatrix}.$$

Additionally, the diagonal-norm nodal bases are

- Gauß-Legendre basis with matrices

$$(51) \quad \underline{\underline{M}} = \begin{pmatrix} 0.555\,555\,555\,555\,555\,4 & 0.0 & 0.0 \\ 0.0 & 0.888\,888\,888\,888\,888\,8 & 0.0 \\ 0.0 & 0.0 & 0.555\,555\,555\,555\,555\,4 \end{pmatrix},$$

$$(52) \quad \underline{\underline{R}} = \begin{pmatrix} 1.478\,830\,557\,701\,236\,2 & -0.666\,666\,666\,666\,666\,5 & 0.187\,836\,108\,965\,430\,5 \\ 1.478\,836\,108\,965\,430\,5 & -0.666\,666\,666\,666\,666\,4 & 1.478\,830\,557\,701\,236 \end{pmatrix},$$

$$(53) \quad \underline{\underline{D}} = \begin{pmatrix} -1.936\,491\,673\,103\,709 & 2.581\,988\,897\,471\,611\,6 & -0.645\,497\,224\,367\,902\,8 \\ -0.645\,497\,224\,367\,902\,6 & -2.465\,190\,328\,815\,662 \times 10^{-31} & 0.645\,497\,224\,367\,902\,6 \\ 0.645\,497\,224\,367\,902\,8 & -2.581\,988\,897\,471\,611\,6 & 1.936\,491\,673\,103\,709 \end{pmatrix}.$$

- Lobatto-Legendre basis with matrices

$$(54) \quad \underline{\underline{M}} = \begin{pmatrix} 0.333\,333\,333\,333\,333\,3 & 0.0 & 0.0 \\ 0.0 & 1.333\,333\,333\,333\,333\,3 & 0.0 \\ 0.0 & 0.0 & 0.333\,333\,333\,333\,333\,3 \end{pmatrix},$$

$$(55) \quad \underline{\underline{R}} = \begin{pmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{pmatrix},$$

$$(56) \quad \underline{\underline{D}} = \begin{pmatrix} -1.5 & 2.0 & -0.5 \\ -0.5 & 0.0 & 0.5 \\ 0.5 & -2.0 & 1.5 \end{pmatrix}.$$

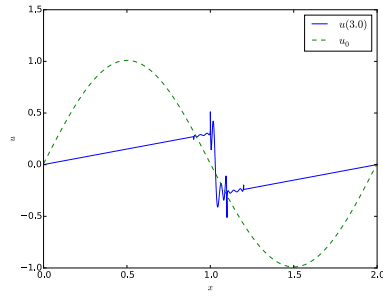
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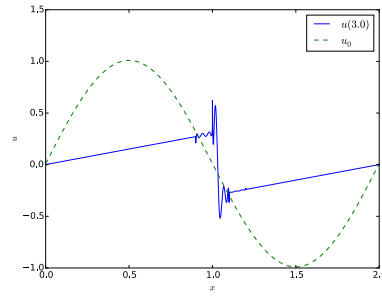
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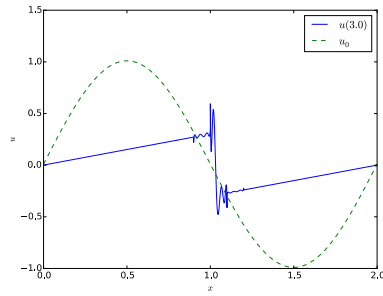
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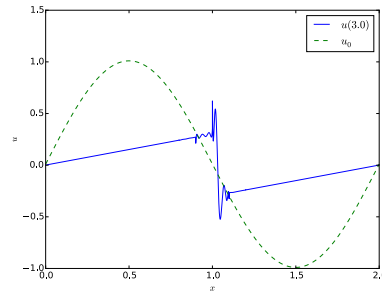
(A) Gauß-Legendre.



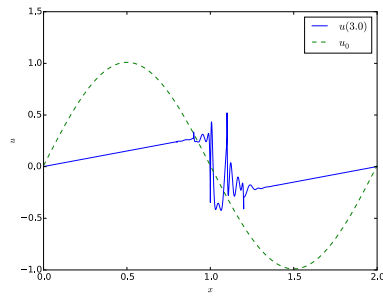
(B) Lobatto-Legendre.



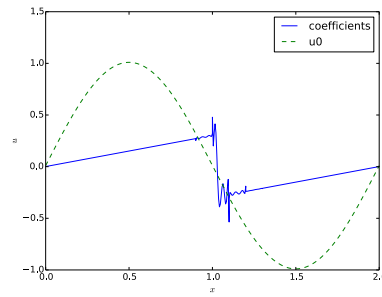
(C) Chebyshev first kind, roots.



(D) Chebyshev first kind, extrema.

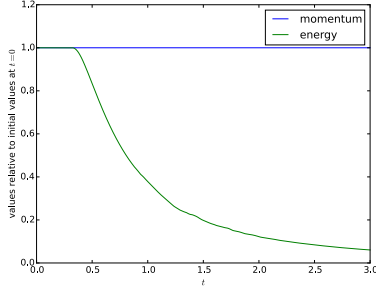


(E) Chebyshev second kind, roots.

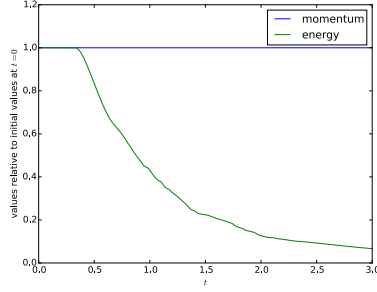


(F) Legendre.

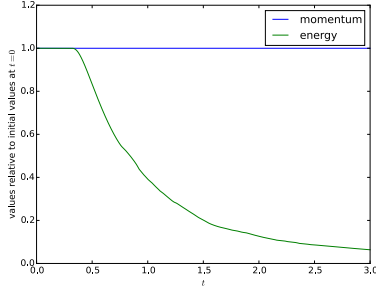
FIGURE 1. Results of the simulations for Burgers' equation using general SBP CPR methods with 20 elements, different bases of order 7 and local Lax-Friedrichs (LLF) flux. Corrections for both divergence and restriction are used. Each Figure shows the values of $u(3)$ (blue) and $u(0) = u_0$ (green) for different bases.



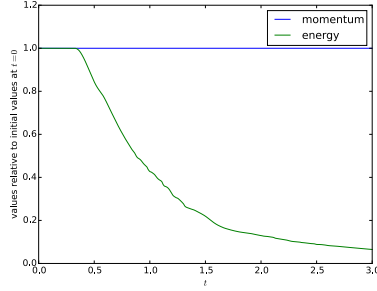
(A) Gauß-Legendre.



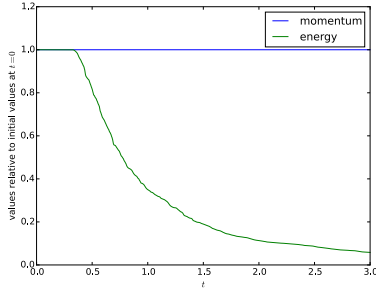
(B) Lobatto-Legendre.



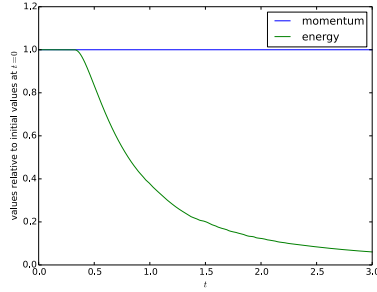
(C) Chebyshev first kind, roots.



(D) Chebyshev first kind, extrema.



(E) Chebyshev second kind, roots.



(F) Legendre.

FIGURE 2. Results of the simulations for Burgers' equation using general SBP CPR methods with 20 elements, different bases of order 7 and local Lax-Friedrichs (LLF) flux. Corrections for both divergence and restriction are used. Each Figure shows the discrete momentum $\underline{1}^T \underline{\underline{M}} \underline{u}$ (blue) and discrete energy $\underline{u}^T \underline{\underline{M}} \underline{u}$ (green) for different bases.